Problem 1.

Compute the following indefinite integrals.

(a) [5pts.]

$$\int \tan^3(x)dx$$

Solution: Using the identity $\tan^2(x) = \sec^2(x) - 1$, we compute

$$\begin{aligned}\int \tan^3(x)dx &= \int \tan(x)(\sec^2(x) - 1)dx \\ &= \int \tan(x)\sec^2(x)dx - \int \tan(x)dx \\ &= \frac{1}{2}\tan^2 x + \int \frac{-\sin x}{\cos x}dx \\ &= \frac{1}{2}\tan^2 x + \ln|\cos x| + C\end{aligned}$$

(b) [5pts.]

$$\int \frac{5dx}{(x-1)(x^2+4)}$$

Solution: The partial fractions decomposition of the integrand is

$$\frac{5dx}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$$

Solving this equation shows that $A = 1$, $B = -1$, and $C = -1$. Therefore the integral may be computed as follows.

$$\begin{aligned}\int \frac{5dx}{(x-1)(x^2+4)} &= \int \frac{dx}{x-1} - \int \frac{x+1}{x^2+4}dx \\ &= \ln|x-1| - \int \frac{xdx}{x^2+4} - \int \frac{dx}{x^2+4} \\ &= \ln|x-1| - \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C \\ &= \ln\left|\frac{x-1}{\sqrt{x^2+4}}\right| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C\end{aligned}$$

Problem 2.

(a) [4pts.] Find M_4 and S_4 for $\int_1^2 \ln x dx$. You do not need to simplify your expressions.

Solution: We see $\Delta x = \frac{2-1}{4} = \frac{1}{4}$. Ergo

$$M_4 = \frac{1}{4}\left[\ln\left(\frac{9}{8}\right) + \ln\left(\frac{11}{8}\right) + \ln\left(\frac{13}{8}\right) + \ln\left(\frac{15}{8}\right)\right]$$
$$S_4 = \frac{1}{12}\left[\ln(1) + 4\ln\left(\frac{5}{4}\right) + 2\ln\left(\frac{3}{2}\right) + 4\ln\left(\frac{7}{4}\right) + \ln(2)\right]$$

(b) [3pts.] Does M_4 overestimate or underestimate the integral?

Solution: We see that $f(x) = \ln x$ is concave down on $[1, 2]$, either by looking at the graph or noticing that $f''(x) = -\frac{1}{x^2} < 0$. Therefore M_4 is an overestimate.

(c) [3pts.] Compute an error bound for S_4.

Solution: Observe that the fourth derivative of $f(x) = \ln x$ is $f^{(4)}(x) = -\frac{6}{x^4}$. Therefore the maximum value of $|f^{(4)}(x)|$ on $[1, 2]$ is $K_4 = 6$. So we compute

$$\begin{aligned}\text{Error}(S_4) &\leq \frac{K_4(2-1)^5}{180(4^4)} \\ &= \frac{6(1)}{180 \cdot 256} \\ &= \frac{1}{30 \cdot 256} \\ &= \frac{1}{7680} \\ &\approx .00013\end{aligned}$$

Problem 3. *8pts.*

A metal plate in the shape of a right isosceles triangle with legs measuring 2 m is submerged vertically in a tank of water with the right angle touching the surface as shown. Calculate the force on one side of the plate. The density of water is 10^3 kg/m^3, and acceleration due to gravity is 9.8 m/s^2.

Solution: The height of the triangle is $\sqrt{2}$ m and the base length of is $2\sqrt{2}$ m. By similar triangles, the width of the triangle at depth y is $f(y) = 2y$. Moreover, the pressure exerted by the water at depth y is $P = \rho g y = 9800y$ Pa. Therefore the following integral calculates the force on one side of the plate.

$$\begin{aligned} 9800\int_0^{\sqrt{2}} f(y)y dy &= 9800\int_0^{\sqrt{2}} 2y^2 dy \\ &= \frac{19600}{3}y^3|_0^{\sqrt{2}} \\ &= \frac{19600\cdot 2\sqrt{2}}{3} \\ &= \frac{39200\sqrt{2}}{3}\text{ N} \\ &\approx 18480\text{ N.} \end{aligned}$$

Problem 4.

Consider the infinite rotational solid generated by rotating the area under the curve $f(x) = \frac{1}{x}$ on $[1, \infty)$ around the x-axis.

(a) [4pts.] Set up (but do not attempt to evaluate!) an improper integral that gives the surface area of this solid.

Solution: We see $f'(x) = -\frac{1}{x^2}$, so

$$\begin{aligned} SA &= 2\pi\int_1^{\infty} f(x)\sqrt{1+f'(x)^2}dx \\ &= 2\pi\int_1^{\infty} \frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx \end{aligned}$$

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(b) [4pts.] Decide whether the integral you found in part (a) converges or diverges.

Solution: We observe that $\frac{1}{x}\sqrt{1+\frac{1}{x^4}} > \frac{1}{x} > 0$, so by the Comparison Test, since $\int_1^{\infty}\frac{dx}{x}$ diverges, so does our surface area integral. Therefore the surface area is infinite.

(c) [4pts.] Recall that an expression for the volume of the trumpet is $V = \pi\int_1^{\infty}\frac{dx}{x^2}$. Does the volume converge or diverge? If it converges, what is it?

Solution: By the p-integral test, the volume converges to $\pi\left(\frac{1^{1-2}}{2-1}\right) = \pi$. Thus the volume is finite. This rotational solid is called Torricelli's Trumpet or the Horn of Gabriel.

Problem 5.

(a) [3pts.] Find the Taylor polynomials $T_n(x)$ for $f(x) = xe^x - e^x$ centered around $x = 0$.

> **Solution:** We compute
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> $$\begin{aligned} f(x) &= xe^x - e^x & f(0) &= -1 \\ f'(x) &= xe^x + e^x - e^x = xe^x & f'(0) &= 0 \\ f''(x) &= xe^x + e^x & f''(0) &= 1 \\ f^{(3)}(x) &= xe^x + 2e^x & f^{(3)}(0) &= 2 \\ f^{(k)}(x) &= xe^x + (k-1)e^x & f^{(k)}(0) &= k-1 \end{aligned}$$
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> We conclude that for $k \geq 2$, the coefficient on the term $(x-0)^k$ in the Taylor polynomials of f is $\frac{f^{(k)}(0)}{k!} = \frac{k-1}{k!} = \frac{1}{k\cdot(k-2)!}$. Ergo the Taylor polynomials $T_n(x)$ are
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> $$T_n(x) = -1 + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4\cdot 2!}x^4 + \cdots \frac{1}{n\cdot(n-2)!}x^n.$$

(b) [4pts.] Find a bound for the error $|T_n(1) - f(1)|$, for $n \geq 1$.

> **Solution:** For $n \geq 1$, $|f^{(n+1)}(x)| = |xe^x + (n)e^x| < (n+1)e = K$ on $[0,1]$; that is, all the derivatives are increasing functions that take their maximum value at 1. So our error bound is
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> $$\begin{aligned} |T_n(1) - f(1)| &\leq K \cdot \frac{|1-0|^{n+1}}{(n+1)!} \\ &= (n+1)e \cdot \frac{1}{(n+1)!} \\ &= \frac{e}{n!} \end{aligned}$$

(c) [3pts.] Find an n such that $|T_n(1) - f(1)| < \frac{1}{100}$. [Hint: Start by slightly increasing the bound you found in part (b) to have an integer in the numerator.]

> **Solution:** Notice that $\frac{e}{n!} < \frac{3}{n!}$, so it suffices to choose n such that $\frac{3}{n!} < \frac{1}{100}$. If $n = 6$, we have $\frac{3}{n!} = \frac{1}{2\cdot 4\cdot 5\cdot 6} = \frac{1}{240} < \frac{1}{100}$.